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On Duality of Multiobjective Fractional Measurable Subset Selection Problems

VASILE PREDA

University of Bucharest, Mathematics Faculty, 14, Academiei Street, Bucharest, Romania

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In this paper we prove some duality results for the multiobjective fractional measurable subset selection problems under more general assumptions than convexity. © 1995 Academic Press, Inc.

1. INTRODUCTION

In this paper let (X, Γ, μ) be a finite atomless measure space with L_1 (X, Γ, μ) separable, and let $f_1, \ldots, f_m, g_1, \ldots, g_m, h_1, \ldots, h_p$ be real-valued n-set functions defined on Γ^n, the n-fold product of a σ-algebra member Γ of subsets of a set X. We assume $g_i(S) > 0$ for all $S \in \Gamma^n$ and each $i \in \mathcal{M} = \{1, 2, \ldots, m\}$. Then we consider a multiobjective fractional programming problem (P) involving vector-valued n-set functions

$$\text{(P) Minimize } (f_1(S)/g_1(S), \ldots, f_m(S)/g_m(S)) \text{ subject to: } h(S) \leqq 0,\ S \in \Gamma^n,$$

where $h = (h_1, \ldots, h_p)$.

The analysis of optimization problems with set functions has been the subject of several papers [4–11, 16, 22, 23, 33, 34]. These problems arise in various mathematical areas [3, 4, 13, 14, 17, 18, 20, 25–30, 32].

For the multiobjective fractional programming problems involving point functions by using the concept of efficiency, in [12, 24] some duality results are stated.

In [1, 2] some classes of functions, called B-vex, pseudo B-vex, and quasi B-vex functions are introduced. Also some sufficient optimality conditions and duality results are obtained for a nonlinear programming problem involving B-vex and B-invex functions.

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In this paper we will define the notions of (ρ, b)-vexity and strict (ρ, b)-vexity for nondifferentiable and differentiable n-set functions and we will also state some duality results for the multiobjective fractional programming problem (P), which involves n-set vectorial functions.

As in [24], we consider a parametric multiobjective problem (P_λ) relative to (P)

$$(P_\lambda) \text{ Minimize } (f_1(S) - \lambda_1 g_1(S), ..., f_m(S) - \lambda_m g_m(S)) \text{ subject to: } h(S) \leqq 0, S \in \Gamma^n,$$

where $\lambda = (\lambda_1, ..., \lambda_m)$ $\lambda_i \in R$, $i \in \mathcal{M}$.

In Section 3, some properties for (ρ, b)-vex functions are given and in Section 4 some duality results are stated between (P_λ) and a Mond–Weir dual problem.

2. Some Definitions and Preliminaries

For $x = (x_1, ..., x_m)$, $y = (y_1, ..., y_m) \in R^m$ we put $x \leqq y$ iff $x_i \leqq y_i$ for each $i \in \mathcal{M}$; $x \leq y$ iff $x_i \leqq y_i$ for each $i \in \mathcal{M}$, with $x \neq y$; $x < y$ iff $x_i < y_i$ for each $i \in \mathcal{M}$. We note that $x \in R^m_+$ iff $x \geqq 0$. Now for $x, y \in R^m$ we put $\langle x, y \rangle = \sum_{i=1}^m x_i y_i$ for the inner product of x and y.

Let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable, and let (Γ^n, d) be a pseudometric space, where d is the pseudometric on Γ^n defined by

$$d(S, T) = \left\{ \sum_{k=1}^{n} [\mu(S_k \triangle T_k)]^2 \right\}^{1/2},$$

$S = (S_1, ..., S_n)$, $T = (T_1, ..., T_n)$, where $\triangle$ denotes the symmetric difference.

For $h \in L_1(X, \Gamma, \mu)$ and $S \in \Gamma$ with indicator characteristic function $I_S \in L_\infty(X, \Gamma, \mu)$, the integral $\int_S h d\mu$ will be denoted by $\langle h, I_S \rangle$.

Now we will define the notion of differentiability for n-set functions. This was originally introduced by Morris [23] for set functions; their n-set counterparts are discussed in Corley [9].

A function $\varphi : \Gamma \to R$ is differentiable at $S^0 \in \Gamma$ if there exists $D\varphi(S^0) \in L_1(X, \Gamma, \mu)$, called the derivative of φ at S^0, and there exists $\psi : \Gamma \times \Gamma \to R$ such that for each $S \in \Gamma$,

$$\varphi(S) = \varphi(S^0) + \langle D\varphi(S^0), I_S - I_{S^0} \rangle + \psi(S, S^0),$$

where $\psi(S, S^0)$ is $0(d(S, S^0))$, that is, $\lim_{d(S,S^0)\to 0} \psi(S, S^0)/d(S, S^0) = 0$.

A function $F : \Gamma^n \to R$ is said to have a partial derivative at $S^0 = (S_1^0, \ldots, S_n^0)$ with respect to its kth argument if the function $\varphi(S_k) = F(S_1^0, \ldots, S_{k-1}^0, S_k, S_{k+1}^0, \ldots, S_n^0)$ has derivative $D\varphi(S_k^0)$ and we define $D_k F(S^0) = D\varphi(S_k^0)$. If there exists $D_k F(S^0)$, $1 \leqq k \leqq n$, we put $DF(S^0) = (D_1 F(S^0), \ldots, D_n F(S^0))$.

The function $F : \Gamma^n \to R$ is differentiable at S^0 if there exists $DF(S^0)$ and $\psi : \Gamma^n \times \Gamma^n \to R$ such that

$$F(S) = F(S^0) + \sum_{k=1}^{n} \langle D_k F(S^0), I_{S_k} - I_{S_k^0} \rangle + \psi(S, S^0),$$

where $\psi(S, S^0)$ is $0(d(S, S^0))$.

In the last part of this section we will give some definitions and preliminary results relative to the problems (P) and (P_λ).

A feasible solution S^0 of (P) is said to be an efficient solution of (P) if there exists no other feasible solution S of (P) such that $f_i(S)/g_i(S) \leqq f_i(S^0)/g_i(S^0)$, for all $i \in \mathcal{M}$, with strict inequality for at least one $i \in \mathcal{M}$.

A feasible solution S^0 of (P) is said to be a weakly efficient solution of (P) if there exists no other feasible solution S of (P) such that $f_i(S)/g_i(S) < f_i(S^0)/g_i(S^0)$ for all $i \in \mathcal{M}$.

From these definitions it is clear that an efficient solution of (P) is also weakly efficient. Also if S^0 is an efficient solution of (P) then there exists $\lambda \in R_+^m$ such that S^0 is an efficient solution of (P_λ).

Now we give some necessary optimality conditions which will be needed in our discussion of duality for problem (P). (See Zalmai [34, Theorems 3.1 and 3.2] and Corley [9, Theorem 3.7].)

LEMMA 2.1. *Let S^0 be a regular efficient solution* (*or weakly efficient solution*) [9, 34] *of* (P) *and assume that f, g, and h are differentiable at S^0. Then there exist $u^0 \in R_+^m$, $\sum_{i=1}^m u_i^0 = 1$, $v^0 \in R_+^p$,* and *$\lambda^0 \in R_+^m$, such that*

$$\left\langle \sum_{i=1}^{m} u_i^0 (D_k f_{iS^0} - \lambda_i^0 D_k g_{iS^0}) + \sum_{j=1}^{p} v_j^0 D_k h_{jS^0}, I_{S_k} - I_{S_k^0} \right\rangle \geqq 0 \tag{1}$$

for all $S_k \in \Gamma$, $1 \leqq k \leqq n$,

$$u_i^0 (f_i(S^0) - \lambda_i^0 g_i(S^0)) \geqq 0, i \in \mathcal{M}; \qquad v_j^0 h_j(S^0) = 0,\ j \in \mathcal{P}. \tag{2}$$

3. (ρ, b)-VEX FUNCTIONS AND SOME PROPERTIES

In this section we will give the notions of (ρ, b)-vexity for n-set functions. Thus we will extend the notion of b-vexity for point functions, defined by Bector [1, 2] and the concept of ρ-convexity for n-set functions, defined by Zalmai [33].

First consider the definition of the Morris sequence. According to [23, Proposition 3.2 and Lemma 3.3], for any $(S, T, \lambda) \in \Gamma \times \Gamma \times [0, 1]$, there exist two sequences $(S^k)_k$ and $(T^k)_k$ in Γ such that

$$I_{S^k} \to \lambda I_{T\setminus S} \quad \text{and} \quad I_{T^k} \to (1-\lambda) I_{S\setminus T} \tag{3}$$

imply

$$I_{S^k \cup T^k \cup (S\cap T)} \to \lambda I_T + (1-\lambda) I_S, \tag{4}$$

where w^* stands for the w^*-convergence. Now, a Morris sequence with (S, T, λ) is a sequence $\{V^k(\lambda) = S^k \cup T^k \cup (S \cap T)\}$ satisfying (3) and (4).

Let $\mathcal{S}$ be a subfamily of Γ^n. $\mathcal{S}$ is a convex set [23] if for any $S, T \in \mathcal{S}$ and $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with (S_i, T_i, λ) for all $i = 1, 2, ..., n$ such that $V^k(\lambda) = (V_1^k, ..., V_n^k) \in \mathcal{S}$, for all $k \in N$. (Here N is the set of natural numbers.)

Let ρ be a real number and $B: \Gamma^n \times \Gamma^n \times [0, 1] \to R_+$ such that $1\text{-}\lambda B(S, T, \lambda) \geqq 0$ for any $S, T \in \Gamma^n$ and $\lambda \in [0, 1]$.

A set function $F: \mathcal{S} \to R$ is called (ρ, B) -vex (resp. strict (ρ, B) -vex) on a convex subfamily $\mathcal{S}$ of Γ^n if for any $S, T \in \mathcal{S}$ (resp. $S \neq T$) and $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}_k$ in Γ associated with (S_i, T_i, λ) for all $i = 1, 2, ..., n$ such that $V^k(\lambda) \in \mathcal{S}$ for all $k \in N$ and

$$\begin{aligned}\overline{\lim_{k\to\infty}} F(V^k(\lambda)) \leqq (<) \lambda B(S, T, \lambda) F(T) + (1 - \lambda B(S, T, \lambda)) F(S) \\ -\rho\lambda(1-\lambda) B(S, T, \lambda) d^2(S, T).\end{aligned} \tag{5}$$

The following result follows from Bector and Singh [1].

Proposition 3.1. *Let $\mathcal{S}$ be a convex subfamily of Γ^n and $F: \mathcal{S} \to R$ a differentiable function at T, where $T \in \mathcal{S}$. If F is a (ρ, B) -vex (strict (ρ, B)-vex) at T, then there exists a function $b: \mathcal{S} \times \mathcal{S} \to R_+$ such that*

$$\begin{aligned}b(S, T)[F(S) - F(T)] \geqq (>) \sum_{k=1}^{n} \langle D_k F_T, I_{S_k} - I_{S_k^0} \rangle \\ + \rho b(S, T) d^2(S, T)\end{aligned} \tag{6}$$

for any $S \in \mathcal{S}$ $(S \neq T)$.

Thus, in the differentiable case, we say that F is (ρ, b)-vex (resp. strict (ρ, b)-vex) at T, if for all $S \in \mathcal{S}$ the inequality (6) holds.

A point $T \in \mathcal{S}$ is a (local) strict minimum for F on $\mathcal{S}$ if $F(T) < F(S)$ for all $S \in \mathcal{S}, S \neq T$ ($S \in \mathcal{S}, S \neq T$ such that $d(S, T) < \varepsilon$, for some $\varepsilon > 0$).

Now the following result connects a point of local minimum and a point of strict minimum for F on $\mathcal{S}$.

PROPOSITION 3.2. *Let $\mathcal{S}$ be a convex subfamily of Γ^n and let $F:\mathcal{S} \to R$ be a strict (ρ, B)-vex function, where $\rho \geqq 0$. Then a point T of local minimum for F on $\mathcal{S}$ is the unique strict minimum for F on $\mathcal{S}$.*

Proof. Since T is a local minimum for F on $\mathcal{S}$ then there exists an $\varepsilon > 0$ such that

$$F(T) < F(S) \qquad \text{for all } S \in \mathcal{S} \tag{7}$$

with $d(S, T) < \varepsilon$. Suppose the contrary, namely, T is the unique strict minimum for F on S. Then there exists $\Omega \in \mathcal{S}$, $\Omega \neq T$ such that

$$F(\Omega) \leqq F(S). \tag{8}$$

Using the convexity of $\mathcal{S}$ and strict (ρ, B)-vexity of F, we obtain that there exists a Morris sequence $\{V_i^k(\lambda)\}_k$ in Γ associated with (T_i, Ω_i, λ) for all $i = 1, 2, \ldots, n$ and $\lambda \in (0, 1)$ such that $V^k(\lambda) \in S$ for all $k \in N$ and

$$\overline{\lim_{k\to\infty}} F(V^k(\lambda)) < \lambda B(\Omega, T, \lambda)F(\Omega) + (1 - \lambda B(\Omega, T, \lambda))F(T)$$
$$-\rho\lambda(1 - \lambda)B(\Omega, T, \lambda)d^2(\Omega, T).$$

Since $\rho \geqq 0$ and $B \geqq 0$ we get

$$\overline{\lim_{k\to\infty}} F(V^k(\lambda)) < \lambda B(\Omega, T, \lambda)F(\Omega) + (1 - \lambda B(\Omega, T, \lambda))F(T). \tag{9}$$

Now consider the following cases.

Case 1. $B(\Omega, T, \lambda) = \lambda^{-1}$. Then by (9) and (8) we obtain

$$\overline{\lim_{k\to\infty}} F(V^k(\lambda)) < F(T). \tag{10}$$

Case 2. $B(\Omega, T, \lambda) = 0$. Then by (9) we also get (10).

Case 3. $B(\Omega, T, \lambda) \neq 0$ and $\neq \lambda^{-1}$. In this case, by (9), (8), and $1 - \lambda B(\Omega, T, \lambda) \geqq 0$, relation (10) also follows.

Using the definition of $d(S, T)$ we obtain $\lim_{k\to\infty} d(V^k(\lambda), T) = \lambda d(\Omega, T)$. Thus there exists $\delta > 0$ and a natural number k_1 such that $d(V^k(\lambda), T) < \varepsilon$ for any $0 < \lambda < \delta$ and $k \geqq k_1$. By (7) we obtain

$$F(T) \leqq F(V^k(\lambda)) \qquad \text{for any } 0 < \lambda < \delta \text{ and } k \geqq k_1. \tag{11}$$

Also by (10), we get that there exists a natural number k_2 such that

$$F(V^k(\lambda)) < F(T) \qquad \text{for } k \geqq k_2. \tag{12}$$

But for k such that $k \geqq k_1$ and $k \geqq k_2$ we obtain that inequality (11) is not compatible with (12). Thus the proposition is proved.

We remark that under the assumptions of Proposition 3.2 we have that a local minimum point for F on $\mathcal{S}$ is also a minimum point for F on $\mathcal{S}$ and a strict local minimum point for F on $\mathcal{S}$ is also a strict minimum point for F on $\mathcal{S}$.

4. Duality

In this section, in the differentiable case, a Mond–Weir dual for (P_λ) is defined and some duality results in (ρ, b)-vexity assumptions are stated. For the problem (P_λ) we consider the dual problem

(D) Maximize $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ subject to

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{m} u_i(D_k f_{iT} - \lambda_i D_k g_{iT}) + \sum_{j=1}^{p} v_j D_k h_{jT}, I_{Sk} - I_{Tk} \right\rangle > 0 \tag{13}$$

for all $S_k \in \Gamma$, $1 \leqq k \leqq n$,

$$u_i(f_i(T) - \lambda_i g_i(T)) \geqq 0, \qquad i \in \mathcal{M} \tag{14}$$

$$v_j h_j(T) \geqq 0, \qquad j \in \mathcal{P}; \tag{15}$$

$$u \in R_+^m, \sum_{i=1}^{m} u_i = 1, v \in R_+^p, \lambda \in R_+^m, \tag{16}$$

where $\mathcal{P} = \{1, 2, \ldots, p\}$. Consider $b : \Gamma^n \times \Gamma^n \to R_+$.

Theorem 4.1 (Weak Duality). *Let (T, u, v, λ) be a feasible solution for* (D) *and assume*

(i1) $f_i(\cdot) - \lambda_i g_i(\cdot)$ *for each $i \in \mathcal{M}$, is (ρ_i, b)-vex;*

(i2) *for each $j \in \mathcal{P}$, h_j is (ρ_j', b)-vex.*

We also assume that any of the following conditions hold:

(i3) $u_i > 0$ *for* $i \in \mathcal{M}$ *and* $\langle u, \rho \rangle + \langle v, \rho' \rangle \geqq 0$ *and* $f_i(\cdot) - \lambda_i g_i(\cdot)$ *is strict* (ρ_i, b)*-vex for some* $i \in \mathcal{M}$;

(i4) $\langle u, \rho \rangle + \langle v, \rho' \rangle > 0$, *where* $\rho = (\rho_1, ..., \rho_m)$, $\rho' = (\rho_1', ..., \rho_m')$.

Then for any S, a feasible solution for (P), *the following cannot hold*

$$f_i(S)/g_i(S) \leqq \lambda_i \qquad \text{for any } i \in \mathcal{M} \tag{17}$$

$$f_j(S)/g_j(S) < \lambda_j \qquad \text{for some } j \in \mathcal{M}. \tag{18}$$

Proof. Let us suppose the contrary, that (17) and (18) hold. Hence there exists S, a feasible solution for (P_λ), such that (17) and (18) hold. Now we consider two cases corresponding respectively to (i3) and (i4).

If hypothesis (i2) holds, then $u_i > 0$ for any $i \in \mathcal{M}$. Thus by (17) and (18) we get

$$\sum_{i=1}^{m} u_i(f_i(S) - \lambda_i g_i(S)) < 0. \tag{19}$$

Using the feasibility of S, and the relations (15) and (16) we have

$$v_j h_j(S) \leqq 0 \leqq v_j h_j(T) \qquad \text{for any } j \in \mathcal{P}. \tag{20}$$

Combining (14), (19), and (20) we obtain

$$\begin{aligned} &\sum_{i=1}^{m} u_i(f_i(S) - \lambda_i g_i(S)) + \sum_{j=1}^{p} v_j h_j(S) \\ &\quad < \sum_{i=1}^{m} u_i(f_i(T) - \lambda_i g_i(T)) + \sum_{j=1}^{p} v_j h_j(T). \end{aligned} \tag{21}$$

On the other hand, from (ρ_i, b)-vexity of $f_i(\cdot) - \lambda_i g_i(\cdot)$, $i \in \mathcal{M}$, given by (i1) and (ρ_j', b)-vexity of h_j, $j \in \mathcal{P}$, given by the assumption (i2), we get

$$\begin{aligned} &b(S, T)[f_i(S) - \lambda_i g_i(S) - (f_i(T) - \lambda_i g_i(T))] \\ &\quad \geqq \sum_{k=1}^{m} \langle D_k f_{iT} - \lambda_i D_k g_{iT}, I_{S_k} - I_{T_k} \rangle + \rho_i d^2(S, T) \end{aligned} \tag{22}$$

for any $i \in \mathcal{M}$, with strict inequality for some i, and

$$b(S, T)[h_j(S) - h_j(T)] \geqq \sum_{k=1}^{n} \langle D_k h_{jT}, I_{S_k} - I_{T_k} \rangle + \rho'_j d^2(S, T) \tag{23}$$

for any $j \in \mathcal{P}$.

Because $u_i > 0$ for $i \in \mathcal{M}$, and $v \geq 0$, by (22) and (23) we obtain

$$\begin{aligned} & b(S, T)\left[\sum_{i=1}^{m} u_i(f_i(S) - \lambda_i g_i(S)) + \sum_{j=1}^{p} v_j h_j(S)\right] \\ & \quad - b(S, T)\left[\sum_{i=1}^{m} u_i(f_i(T) - \lambda_i g_i(T)) + \sum_{j=1}^{p} v_j h_j(T)\right] \\ & > \sum_{k=1}^{n} \left\langle \sum_{i=1}^{m} u_i(D_k f_{iT} - \lambda_i D_k g_{iT}) \right. \\ & \quad \left. + \sum_{j=1}^{p} v_j D_k h_{jT}, I_{S_k} - I_{T_k} \right\rangle + \langle u, \rho \rangle + \langle v, \rho' \rangle. \end{aligned} \tag{24}$$

Using $b \geqq 0$ and (21), by (24) we get

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{m} u_i(D_k f_{iT} - \lambda_i D_k g_{iT}) + \sum_{j=1}^{p} v_j D_k h_{jT}, I_{S_k} - I_{T_k} \right\rangle + \langle u, \rho \rangle + \langle v, \rho' \rangle < 0.$$

Since we have $\langle u, \rho \rangle + \langle v, \rho' \rangle \geqq 0$, given by (i3) and (13), we obtain a contradiction. Thus, in case (i3), the theorem is proved.

Consider now case (i4). We proceed as in case (i3) and then from (17), (18), (20), and $u \geq 0$, we obtain

$$\begin{aligned} & \sum_{i=1}^{m} u_i(f_i(S) - \lambda_i g_i(S)) + \sum_{j=1}^{p} v_j h_j(S) \\ & \quad \leqq \sum_{i=1}^{m} u_i(f_i(T) - \lambda_i g_i(T)) + \sum_{j=1}^{p} v_j h_j(T). \end{aligned} \tag{25}$$

We also have (22) and (23). Since $u \in R_+^m$, $v \in R_+^p$, by (25) we get

$$\sum_{k=1}^{n} \left\langle \sum_{i=1}^{m} u_i(D_k f_{iT} - \lambda_i D_k g_{iT}) + \sum_{j=1}^{p} v_j D_k h_{jT}, I_{S_k} - I_{T_k} \right\rangle + \langle u, \rho \rangle + \langle v, \rho' \rangle \leqq 0.$$

From this inequality, (13) and assumption (i4) we obtain a contradiction. Thus the theorem is proved.

Remark 4.1. In case (i3) it is possible to use the strict (ρ, b)-vexity because $S \neq T$ (if $S = T$, since $u_i > 0$ for any $i \in \mathcal{M}$, S is feasible and (14) we obtain a contradiction with (17) and (18)).

COROLLARY 4.1. *Suppose that Theorem* 4.1 *holds and* S^0, $(S^0, u^0, v^0, \lambda^0)$ *are feasible solutions respectively for* (P_{λ^0}) *and* (D). *Then* S^0 *is an efficient solution for* (P_{λ^0}) *and* $(S^0, u^0, v^0, \lambda^0)$ *is an efficient solution for* (D).

Proof. Assume the contrary. If S^0 is not an efficient solution for (P_{λ^0}) then there exists a feasible solution S' for (P_{λ^0}) such that

$$f_i(S') \leqq \lambda_i^0 g_i(S') \qquad \text{for any } i \in \mathcal{M} \tag{26}$$

and

$$f_j(S') \leqq \lambda_j^0 g_j(S') \qquad \text{for some } j \in \mathcal{M}. \tag{27}$$

Since $(S^0, u^0, v^0, \lambda^0)$ is a feasible solution for (D) by (26), (27), and Theorem 4.1 we obtain a contradiction. Hence S^0 is an efficient solution for (P_{λ^0}). In the same way we obtain that $(S^0, u^0, v^0, \lambda^0)$ is an efficient solution for (D).

Now we have the following strong duality result.

THEOREM 4.2 (Strong Duality). *Let* S^0 *be a regular efficient solution for* (P). *Then there exist* $u^0 \in R_+^m$, $\sum_{i=1}^m u_i^0 = 1$, $v^0 \in R_+^p$, and $\lambda^0 \in R_+^m$, *such that* $(S^0, u^0, v^0, \lambda^0)$ *is a feasible solution for* (D).

Moreover, if a weak duality result between (P_{λ^0}) *and* (D) *holds* (*for example Theorem* 4.1 *holds*) *then* $(S^0, u^0, v^0, \lambda^0)$ *is an efficient solution for* (D).

Proof. Using Lemma 2.1 we obtain that there exist $u^0 \in R_+^m$, $\sum_{i=1}^m u_i^0 = 1$, $v^0 \in R_+^p$, and (1) and (2) hold. Thus we see that $(S^0, u^0, v^0, \lambda^0)$ satisfies (13)–(16). Hence $(S^0, u^0, v^0, \lambda^0)$ is a feasible solution for (D). Moreover if Theorem 4.1 holds then, by Corollary 4.1 we obtain that this solution $(S^0, u^0, v^0, \lambda^0)$ is also an efficient solution for (D).

Now we establish a Mangasarian type [19] strict converse theorem for (P_λ) and (D).

THEOREM 4.3. (Strict Converse Duality). *Let* S^* *and* $(S^0, u^0, v^0, \lambda^0)$ *be efficient solutions respectively for* (P_{λ^0}) *and* (D). *We suppose*

- (j1) $\sum_{i=1}^m u_i^0(f_i(S^*) - \lambda_i^0 g_i(S^*)) \leqq \sum_{i=1}^m u_i^0(f_i(S^0) - \lambda_i^0 g_i(S^0))$;
- (j2) *for any* $i \in \mathcal{M}$, $f_i(\cdot) - \lambda_i^0 g_i(\cdot)$ is strictly (ρ_i, b)-*vex*;
- (j3) *for any* $j \in \mathcal{P}$, $h_j(\cdot)$ *is* (ρ_j', b)-*vex*;
- (j4) $\langle u^0, \rho\rangle + \langle v^0, \rho'\rangle \geq 0$.

Then $S^0 = S^*$.

Proof. Suppose $S^0 \neq S^*$. Using (j2) and (j3) we obtain

$$\begin{aligned} &b(S^*, S^0)[f_i(S^*) - \lambda_i^0 g_i(S^*) - (f_i(S^0) - \lambda_i^0 g_i(S^0))] \\ &\quad > \sum_{k=1}^{n} \langle D_k f_{iS^0} - \lambda_i^0 D_k g_{iS^0}, I_{S_k^*} - I_{S_k^0}\rangle + \rho_i d^2(S^*, S^0) \end{aligned} \tag{28}$$

for any $i \in \mathcal{M}$, and

$$b(S^*, S^0)[h_j(S^*) - h_j(S^0)] \geqq \sum_{k=1}^{n} \langle D_k h_{jS^0}, I_{S_k^*} - I_{S_k^0}\rangle + \rho_j' d^2(S^*, S^0). \tag{29}$$

Since $u^0 \in R_+^m$, $\sum_{i=1}^m u_i^0 = 1$, and $v^0 \in R_+^p$, from (28) and (29) we obtain

$$\begin{aligned} &b(S^*, S^0)\left\{\sum_{i=1}^{m} u_i^0[f_i(S^*) - \lambda_i^0 g_i(S^*) - (f_i(S^0) - \lambda_i^0 g_i(S^0))]\right. \\ &\quad \left. + \sum_{j=1}^{p} v_j^0(h_j(S^*) - h_j(S^0))\right\} \\ &> \sum_{k=1}^{n} \left\langle \sum_{i=1}^{m} u_i^0(D_k f_{iS^0} - \lambda_i^0 D_k g_{iS^0}) + \sum_{j=1}^{p} v_j^0 D_k h_{jS^0}, I_{S_k^*} - I_{S_k^0}\right\rangle \\ &\quad + (\langle u^0, \rho\rangle + \langle v^0, \rho'\rangle)\, d^2(S^*, S^0). \end{aligned}$$

Now, because $(S^0, u^0, v^0, \lambda^0)$ is a feasible solution for (D), by (13) and (j4) we get

$$\begin{aligned} &b(S^*, S^0)\left\{\sum_{i=1}^{m} u_i^0[f_i(S^*) - \lambda_i^0 g_i(S^*) - (f_i(S^0) - \lambda_i^0 g_i(S^0))]\right. \\ &\quad \left. + \sum_{j=1}^{p} v_j^0(h_j(S^*) - h_j(S^0))\right\} > 0. \end{aligned} \tag{30}$$

Since $v_j^0 h_j(S^*) \leqq 0$ and $v_j^0 h_j(S^0) \geqq 0$ for any $j \in \mathcal{P}$ and $b \geqq 0$, by (30) we obtain

$$b(S^*, S^0)\left\{\sum_{i=1}^{m} u_i^0\,[f_i(S^*) - \lambda_i^0 g_i(S^*) - (f_i(S^0) - \lambda_i^0 g_i(S^0))]\right\} > 0$$

which contradicts the assumption (j1). Thus the theorem is proved.

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